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# To Mock a Mockingbird

A certain enchanted forest is inhabited by talking birds. Given any birds A and B, if you call out the name of B to A, then A will respond by calling out the name of some bird to you; this bird we designate by AB. Thus AB is the bird named by A upon hearing the name of B. Instead of constantly using the cumbersome phrase "A's response to hearing the name of B," we shall more simply say: "A's response to B." Thus AB is A's response to B. In general, A's response to B is not necessarily the same as B's response to A—in symbols, AB is not necessarily the same bird as BA. Also, given three birds A, B, and C, the bird A(BC) is not necessarily the same as the bird (AB)C. The bird A(BC) is A's response to the bird BC, whereas the bird (AB)C is the response of the bird AB to the bird C. The use of parentheses is thus necessary to avoid ambiguity; if I just wrote ABC, you could not possibly know whether I meant the bird A(BC) or the bird (AB)C.

*Mockingbirds:* By a *mockingbird* is meant a bird M such that for any bird x, the following condition holds:

$$Mx = xx$$

M is called a mockingbird for the simple reason that its response to any bird x is the same as x's response to itself— in other words, M *mimics* x as far as its response to x goes. This means that if you call out x to M or if you call out x to itself, you will get the same response in either case.*

*Composition:* The last technical detail before the fun starts

* For handy reference to the birds, each is alphabetically listed in "Who's Who Among the Birds," p. 244.

is this: Given any birds A, B, and C (not necessarily distinct) the bird C is said to *compose* A with B if for every bird x the following condition holds:

$$Cx = A(Bx)$$

In words, this means that C's response to x is the same as A's response to B's response to x.

# TO MOCK A MOCKINGBIRD

## 1 • The Significance of the Mockingbird

It *could* happen that if you call out B to A, A might call the same bird B back to you. If this happens, it indicates that A is *fond* of the bird B. In symbols, A is fond of B means that $AB = B$.

We are now given that the forest satisfies the following two conditions.

$C_1$ *(the composition condition):* For any two birds A and B (whether the same or different) there is a bird C such that for any bird x, $Cx = A(Bx)$. In other words, for any birds A and B there is a bird C that composes A with B.

$C_2$ *(the mockingbird condition):* The forest contains a mockingbird M.

One rumor has it that every bird of the forest is fond of at least one bird. Another rumor has it that there is at least one bird that is not fond of any bird. The interesting thing is that it is possible to settle the matter completely by virtue of the given conditions $C_1$ and $C_2$.

Which of the two rumors is correct?

*Note:* This is a basic problem in the field known as *combinatory logic*. The solution, though not lengthy, is extremely ingenious. It is based on a principle that derives ultimately from the work of the logician Kurt Gödel. This principle will permeate parts of many of the chapters that follow.

## 2 • Egocentric?

A bird x is called *egocentric* (sometimes *narcissistic*) if it is fond of itself—that is, if x's response to x is x. In symbols, x is egocentric if $xx = x$.

The problem is to prove that under the given conditions $C_1$ and $C_2$ of the last problem, at least one bird is egocentric.

## 3 • Story of the Agreeable Bird

Two birds A and B are said to *agree* on a bird x if their responses to x are the same—in other words if $Ax = Bx$. A bird A is called *agreeable* if for every bird B, there is at least one bird x on which A and B agree. In other words, A is *agreeable* if for every bird B there is a bird x such that $Ax = Bx$.

We now consider the following variant of Problem 1: We are given the composition condition $C_1$, but we are not given that there is a mockingbird; instead, we are given that there is an agreeable bird A. Is this enough to guarantee that every bird is fond of at least one bird?

*A bonus question:* Why is Problem 1 nothing more than a special case of Problem 3? *Hint:* Is a mockingbird necessarily agreeable?

## 4 • A Question on Agreeable Birds

Suppose that the composition condition $C_1$ of Problem 1 holds and that A, B, and C are birds such that C composes A with B. Prove that if C is agreeable then A is also agreeable.

## 5 • An Exercise in Composition

Again suppose that condition $C_1$ holds. Prove that for any birds A, B, and C there is a bird D such that for every bird x, $Dx = A(B(Cx))$. This fact is quite useful.

## 6 • Compatible Birds

Two birds A and B, either the same or different, are called *compatible* if there is a bird x and a bird y, either the same or different, such that Ax = y and By = x. This means that if you call out x to A then you will get y as a response, whereas if you call out y to B, you will get x as a response.

Prove that if conditions $C_1$ and $C_2$ of Problem 1 hold, then any two birds A and B are compatible.

## 7 • Happy Birds

A bird A is called *happy* if it is compatible with itself. This means that there are birds x and y such that Ax = y and Ay = x.

Prove that any bird that is fond of at least one bird must be a happy bird.

## 8 • Normal Birds

We will henceforth call a bird *normal* if it is fond of at least one bird. We have just proved that every normal bird is happy. The converse is not necessarily true; a happy bird is not necessarily normal.

Prove that if the composition condition $C_1$ holds and if there is at least one happy bird in the forest, then there is at least one normal bird.

# HOPELESS EGOCENTRICITY

## 9 • Hopelessly Egocentric

We recall that a bird B is called *egocentric* if BB = B. We call a bird B *hopelessly egocentric* if for *every* bird x, Bx = B. This means that whatever bird x you call out to B is irrelevant; it only calls B back to you! Imagine that the bird's name is Ber-

trand. When you call out "Arthur," you get the response "Bertrand"; when you call out "Raymond," you get the response "Bertrand"; when you call out "Ann," you get the response "Bertrand." All this bird can ever think of is itself!

More generally, we say that a bird A is *fixated* on a bird B if for every bird x, $Ax = B$. That is, all A can think of is B! Then a bird is hopelessly egocentric just in the case that it is fixated on itself.

A bird K is called a *kestrel* if for any birds x and y, $(Kx)y = x$. Thus if K is a kestrel, then for every bird x, the bird Kx is fixated on x.

Given conditions $C_1$ and $C_2$ of Problem 1, and the existence of a kestrel K, prove that at least one bird is hopelessly egocentric.

## 10 • Fixation

If x is fixated on y, does it necessarily follow that x is fond of y?

## 11 • A Fact About Kestrels

Prove that if a kestrel is egocentric, then it must be hopelessly egocentric.

## 12 • Another Fact About Kestrels

Prove that for any kestrel K and any bird x, if Kx is egocentric then K must be fond of x.

## 13 • A Simple Exercise

Determine whether the following statement is true or false: If a bird A is hopelessly egocentric, then for any birds x and y, $Ax = Ay$.

## 14 • Another Exercise

If A is hopelessly egocentric, does it follow that for any birds x and y, $(Ax)y = A$?

## 15 • Hopeless Egocentricity Is Contagious!

Prove that if A is hopelessly egocentric, then for every bird x, the bird Ax is also hopelessly egocentric.

## 16 • Another Fact About Kestrels

In general, it is not true that if $Ax = Ay$ then $x = y$. However, it *is* true if A happens to be a kestrel K. Prove that if $Kx = Ky$ then $x = y$. (We shall henceforth refer to this fact as the *left cancellation law for kestrels*.)

## 17 • A Fact About Fixation

It is possible that a bird can be fond of more than one bird, but it is not possible for a bird to be fixated on more than one bird. Prove that it is impossible for a bird to be fixated on more than one bird.

## 18 • Another Fact About Kestrels

Prove that for any kestrel K and any bird x, if K is fond of Kx, then K is fond of x.

## 19 • A Riddle

Someone once said: "Any egocentric kestrel must be extremely lonely!" Why is this true?

# IDENTITY BIRDS

A bird I is called an *identity* bird if for every bird x the following condition holds:

$$Ix = x$$

The identity bird has sometimes been maligned, owing to the fact that whatever bird x you call to I, all I does is to echo

x back to you. Superficially, the bird I appears to have no intelligence or imagination; all it can do is repeat what it hears. For this reason, in the past, thoughtless students of ornithology referred to it as the *idiot* bird. However, a more profound ornithologist once studied the situation in great depth and discovered that the identity bird is in fact highly intelligent! The *real* reason for its apparently unimaginative behavior is that it has an unusually large heart *and hence is fond of every bird!* So when you call x to I, the reason it responds by calling back x is not that it can't think of anything else; it's just that it wants you to know that it is fond of x!

Since an identity bird is fond of every bird, then it is also fond of itself, so every identity bird is egocentric. However, its egocentricity doesn't mean that it is any more fond of itself than of any other bird!

Now for a few simple problems about identity birds.

### • 20 •

Supposing we are told that the forest contains an identity bird I and that I is agreeable, in the sense of Problem 3. Does it follow that every bird must be fond of at least one bird? *Note:* We are no longer given conditions $C_1$ and $C_2$.

### • 21 •

Suppose we are told that there is an identity bird I and that every bird is fond of at least one bird. Does it necessarily follow that I is agreeable?

### • 22 •

Suppose we are told that there is an identity bird I, but we are not told whether I is agreeable or not. However, we are told that every pair of birds is compatible, in the sense of Problem 6. Which of the following conclusions can be validly drawn?

1.  Every bird is normal—i.e., fond of at least one bird.
2.  I is agreeable.

## 23 • Why?

The identity bird I, though egocentric, is in general not *hope-lessly* egocentric. Indeed, if there were a hopelessly egocentric identity bird, the situation would be quite sad. Why?

# LARKS

A bird L is called a *lark* if for any birds x and y the following holds:

$$(Lx)y = x(yy)$$

Larks have some interesting properties, as we will now see.

### • 24 •

Prove that if the forest contains a lark L and an identity bird I, then it must also contain a mockingbird M.

### • 25 •

One reason I like larks is this: If there is a lark in the forest, then it follows without further ado that every bird is fond of at least one bird. And so you see, the lark has a wonderful effect on the forest as a whole; its presence makes every bird normal. And since all normal birds are happy, by Problem 7, then a lark L in the forest causes all the birds to be happy!

Why is this true?

## 26 • Another Riddle

Why is a hopelessly egocentric lark unusually attractive?

### • 27 •

Assuming that no bird can be both a lark and a kestrel—as any ornithologist knows!—prove that it is impossible for a lark to be fond of a kestrel.